SOUND WAVES IN A MAGNETIZABLE MEDIUM

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This article considers the propagation of small perturbations in a medium which can be inhomogeneously and isotropically magnetized under the action of an electromagnetic field. It is shown that in such a medium there is the possibility of sound waves of the same kind as in a medium with a constant magnetic susceptibility. However, the phase velocities of fast and slow magnetosonic waves can take on imaginary values so that, in strong magnetic fields, there may arise the phenomenon of instability. Investigations were made of the diagrams of the phase velocities for para- and diamagnetic substances for a medium with magnetic saturation; the case of an incompressible medium is discussed.

The equations of motion of a medium which can be inhomogeneously and isotropically magnetized in an electromagnetic field can be written in the form [1]

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0$$

$$\rho T \frac{d}{dt}\left(s + \frac{1}{\rho}\int_0^H \left(\frac{\partial M}{\partial T}\right)_{\rho, H} dH\right) = \tau_{ik}\frac{\partial v_i}{\partial x_k} + \operatorname{div}(\lambda \nabla T) + \frac{\nu_m}{4\pi}(\operatorname{rot}\mathbf{H})^2$$

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla(p+\psi) + \frac{1}{4\pi}(\operatorname{rot}\mathbf{H}\times\mathbf{B}) + M\nabla H + \eta_1\Delta\mathbf{v} + \left(\eta_2 + \frac{1}{3}\eta_1\right)\nabla\operatorname{div}\mathbf{v}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}(\mathbf{v}\times\mathbf{B}) - \nu_m \operatorname{rot}\operatorname{rot}\mathbf{H}$$

$$\operatorname{div}\mathbf{B} = 0 \qquad \left(\psi = \int_0^H \left[M - \rho\left(\frac{\partial M}{\partial \rho}\right)_{T, H}\right] dH\right) \tag{1}$$

Here τ_{ik} is the tensor of the viscous stresses; η_1, η_2 are the constant coefficients of the primary and secondary viscosity; $\nu_m = c^2/4\pi\sigma$ is the magnetic viscosity; it is assumed that $\mathbf{B} = \mathbf{H} + 4\pi M(\rho, T, H)$; here, the function of the magnetization

$$M(\rho, T, H) \equiv \frac{H}{4\pi}[\mu(\rho, T, H) - 1]$$

is assumed to be known (μ is the magnetic permeability of the medium), and $\mathbf{M} \parallel \mathbf{H}$.

Let us consider the propagation of small perturbations in such a medium.

Let the unperturbed state of the medium be characterized by constant values of its parameters ρ_0, $\mathbf{v}_0$, T_0, s_0, p_0, $\mathbf{B}_0$

$$\left(\frac{\partial p}{\partial \rho}\right)_0 \equiv p_\rho, \quad \left(\frac{\partial p}{\partial s}\right)_0 \equiv p_s, \quad \left(\frac{\partial T}{\partial \rho}\right)_0 \equiv T_\rho, \quad \left(\frac{\partial T}{\partial s}\right)_0 \equiv T_s$$

$$\left(\frac{\partial \mu}{\partial H}\right)_0 \equiv \mu_H, \quad \left(\frac{\partial \mu}{\partial \rho}\right)_0 \equiv \mu_\rho, \quad \left(\frac{\partial \mu}{\partial T}\right)_0 \equiv \mu_T$$

[it is assumed that the equation of state of the medium may be written in the forms $p = p(\rho, s)$, $T = T(\rho, s)$].

We select the system of reckoning in such a way that $\mathbf{v}_0 = 0$ and the vector $\mathbf{B}_0$ lies in the plane xy.

Plane waves are being considered, so that

$$\rho = \rho_0 + \rho'(x, t), \quad \mathbf{v} = \mathbf{v}'(x, t), \quad T = T_0 + T'(x, t), \ldots \tag{2}$$

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Here, the squares of the perturbations ρ', $\mathbf{v}'$, T' will be everywhere neglected.

Then, from the third and fourth equation of system (1) we can obtain $B_x = B_{0x}$.

The field $\mathbf{H} = \mathbf{B}/\mu(\rho, T, H)$ in the medium receives a perturbation due to the inhomogeneity of the magnetization, so that, in the approximation under consideration, this perturbation will be a linear function of the perturbations ρ', T, B'. Taking into consideration that $\mathbf{H}_0 = \mathbf{B}_0/\mu_0$ lies in the plane xy, we obtain

$$
\begin{aligned}
H_x' &= -\frac{B_x\mu_\rho}{\mu^2+\mu_H B}\rho' - \frac{B_x\mu_T}{\mu^2+\mu_H B}T' - \frac{B_x B_y \mu_H}{\mu B(\mu^2+\mu_H B)}B_y' \\
H_y' &= -\frac{B_y\mu_\rho}{\mu^2+\mu_H B}\rho' - \frac{B_y\mu_T}{\mu^2+\mu_H B}T' + \frac{\mu^2 B+\mu_H B_x^2}{\mu B(\mu^2+\mu_H B)}B_y' \\
H_z' &= B_z'/\mu_0
\end{aligned}
\tag{3}
$$

Here and in what follows the subscript zero of quantities characterizing the unperturbed state of the medium is omitted.

We note that the dependence of the perturbation H' on ρ', T', and B' was not taken into consideration in [2], where waves in an incompressible ferromagnetic liquid were discussed. In [3], sound waves were discussed starting from a system of equations in which no account was taken of the magnetocaloric effect.

Substituting expressions (2) and (3) into system (1) and discarding the squares of the perturbations, we obtain equations for the perturbed motion which we write in the form

$$
\frac{\partial u_i}{\partial t} + x_{ik}\frac{\partial u_k}{\partial x} = d_{ik}\frac{\partial^2 u_k}{\partial x^2} \, . \tag{4}
$$

Here

$$
u_1 \equiv \rho',\ u_2 \equiv s',\ u_3 = v_y',\ u_4 = v_x',\ u_5 = B_y',\ u_6 = v_z',\ u_7' = B_z'
$$

are not equal to zero, and the components of the matrices $\|x_{ik}\|$ and $\|d_{ik}\|$ have the following values:

$$
\begin{gathered}
x_{14} = \rho, \quad x_{23} = \mu\mu_T m B_x B_y [1 + T_s(s_T^* - m\mu_T^2 B^2)]^{-1} \\
x_{24} = [\mu_T \rho m B^2(\mu_\rho + \mu_T T_\rho) - \mu\mu_T m B_y^2 - \rho(s_\rho^* + s_T^* T_\rho)][1 + T_s(s_T^* - m\mu_T^2 B^2)]^{-1} \\
x_{31} = \mu m(\mu_\rho + \mu_T T_\rho)B_x B_y, \quad x_{32} = \mu m \mu_T T_s B_x B_y \\
x_{35} = -m(\mu^2 + \mu_H B_x^2/B)B_x \\
x_{41} = (p_\rho + \psi_\rho + \psi_T T_\rho)\rho^{-1} + m(\mu_\rho + \mu_T T_\rho)(\rho\mu_\rho B^2 - \mu B_y^2) \\
x_{42} = (p_s + \psi_T T_s)\rho^{-1} + m\mu_T T_s(\rho\mu_\rho B^2 - \mu B_y^2) \\
x_{45} = mB_y(\mu^2 + \mu_H B_x^2/B - \mu\rho\mu_\rho) \\
x_{53} = -B_x, \quad x_{54} = B_y, \quad x_{67} = -B_x/4\pi\rho\mu, \quad x_{76} = -B_x \\
d_{12} = [\lambda T_\rho/\rho T + 4\pi\rho m^2\mu^2\mu_T\nu_m B_y^2(\mu_\rho + \mu_T T_\rho)][1 + T_s(s_T^* - \mu_T^2 m B^2)]^{-1} \\
d_{22} = [\lambda T_s/\rho T + 4\pi\rho m^2\mu^2\mu_T^2\nu_m T_s B_y^2][1 + T_s(s_T^* - m\mu_T^2 B^2)]^{-1} \\
d_{25} = -4\pi\rho m^2\mu\mu_T\nu_m B_y(\mu^2 + \mu_H B_x^2/B)[1 + T_s(s_T^* - m\mu_T^2 B^2)]^{-1} \\
d_{33} = \eta_1/\rho, \quad d_{44} = \eta_2/\rho + 4\eta_1/3\rho \\
d_{51} = -4\pi\rho\mu\nu_m B_y m(\mu_\rho + T_\rho\mu_T), \qquad d_{52} = -4\pi\rho\mu\nu_m m B_y\mu_\rho T_s \\
d_{55} = 4\pi\rho\nu_m m(\mu^2 + \mu_H B_x^2/B), \qquad d_{66} = \eta_1/\rho, \ d_{77} = \nu_m/\mu
\end{gathered}
\tag{5}
$$

where

$$
m = [4\pi\rho\mu(\mu^2 + \mu_H B)]^{-1}
$$

$$
s^* = \frac{1}{\rho}\int_0^H \left(\frac{\partial M}{\partial T}\right)_{\rho,H} dH = \frac{1}{4\pi\rho}\int_0^H \left(\frac{\partial\mu}{\partial T}\right)_{\rho,H} H\,dH, \quad s_T^* = \left(\frac{\partial s^*}{\partial T}\right)_0, \quad s_\rho^* = \left(\frac{\partial s^*}{\partial \rho}\right)_0 .
$$

In what follows, we shall consider small perturbations without taking account of dissipation, i.e., we shall assume $d_{ik} = 0$. Seeking the solution of system (4) in the form

$$
u_i = u_i^\circ \exp[i(kx - \omega t)],
$$

we obtain the result that the phase velocity of a wave $\lambda = \omega/k$ is an eigennumber of the matrix $\|x_{ik}\|$, while the amplitudes u_i° in this wave are proportional to the right-hand eigen vector $\mathbf{r}^{(\lambda)} = (r_1^{(\lambda)}, r_2^{(\lambda)}, \ldots, r_7^{(\lambda)})$, corresponding to a given value of λ ($x_{ik} r_k^{(\lambda)} = \lambda r_i^{(\lambda)}$). The values of λ are the roots of the characteristic equation

$$|x_{ik} - \lambda\delta_{ik}| = 0 \tag{6}$$

and $r_i^{(\lambda)}$ are proportional to the corresponding subdeterminants of the matrix $\| x_{ik} - \lambda\delta_{ik} \|$.

Expanding the determinant (6), we obtain the characteristic equation of system (4) in the form

$$\lambda (\lambda^2 - B_x^2 / 4\pi\rho\mu) \ (\lambda^4 - 2C_1\lambda^2 + C_2) = 0 \tag{7}$$

where

$$2\, C_1 = B_y x_{45} - B_x x_{35} + \rho x_{41} + x_{23} x_{32} + x_{24} x_{42} \tag{8}$$

$$C_2 = \rho B_x \begin{vmatrix} x_{31} x_{35} \\ x_{41} x_{45} \end{vmatrix} - \rho x_{23} \begin{vmatrix} x_{31} x_{32} \\ x_{41} x_{42} \end{vmatrix} + (x_{23} B_y + x_{24} B_x) \begin{vmatrix} x_{32} x_{35} \\ x_{42} x_{45} \end{vmatrix} .$$

Equation (7) has seven roots which give the phase velocities of the following waves:

1) $\lambda^{(1)} = 0$ is an entropy wave, not propagating with respect to the medium; the corresponding right-hand eigenvector has the form

$$r^{(1)} = \left(\begin{vmatrix} x_{32} x_{35} \\ x_{42} x_{45} \end{vmatrix} , \begin{vmatrix} x_{35} x_{31} \\ x_{45} x_{41} \end{vmatrix} , 0, 0, \begin{vmatrix} x_{31} x_{32} \\ x_{41} x_{42} \end{vmatrix} , 0, 0 \right) . \tag{9}$$

2) $\lambda^{(2,3)} = \pm B_x / \sqrt{4\pi\rho\mu} \equiv \pm A_x$ are two Alfvén waves; the right-hand eigenvector corresponding to the Alfvén waves has the form

$$r^{(2,3)} = (0, 0, 0, 0, 0, 1, - B_x / \lambda^{(2,3)}) .$$

3) $\lambda_+^{(4,5)} = \pm\sqrt{C_1 + (C_1^2 - C_2)^{1/2}}$ are two fast magnetosonic waves in a magnetic substance, and

$$\lambda_-^{(6,7)} = \pm \sqrt{C_1 - (C_1^2 - C_2)^{1/2}}$$

are two slow magnetosonic waves in a magnetic substance.

The right-hand eigenvectors corresponding to these waves have the form

$$r^{(4,5)} = r\,(\lambda_+^{(4,5)}), \qquad r^{(6,7)} = r\,(\lambda_-^{(6,7)})$$

where the vector-function $\mathbf{r}(\lambda)$ has the components

$$\begin{aligned}
r_1(\lambda) &= \rho \left(x_{41}\lambda^2 - B_x \begin{vmatrix} x_{31} x_{35} \\ x_{41} x_{45} \end{vmatrix} + x_{23} \begin{vmatrix} x_{31} x_{32} \\ x_{41} x_{42} \end{vmatrix} \right) \\
r_2(\lambda) &= \lambda^2 (x_{23} x_{31} + x_{24} x_{41}) - (B_x x_{24} + B_y x_{23}) \begin{vmatrix} x_{31} x_{35} \\ x_{41} x_{45} \end{vmatrix} \\
r_3(\lambda) &= \lambda \left(\lambda^2 x_{31} - B_y \begin{vmatrix} x_{31} x_{35} \\ x_{41} x_{45} \end{vmatrix} - x_{24} \begin{vmatrix} x_{31} x_{32} \\ x_{41} x_{42} \end{vmatrix} \right) \\
r_4(\lambda) &= \lambda \left(\lambda^2 x_{41} - B_x \begin{vmatrix} x_{31} x_{35} \\ x_{41} x_{45} \end{vmatrix} + x_{23} \begin{vmatrix} x_{31} x_{32} \\ x_{41} x_{42} \end{vmatrix} \right) \\
r_5(\lambda) &= \lambda^2 \begin{vmatrix} B_y B_x \\ x_{31} x_{41} \end{vmatrix} + (B_x x_{24} + B_y x_{23}) \begin{vmatrix} x_{31} x_{32} \\ x_{41} x_{42} \end{vmatrix} \\
r_6(\lambda) &= r_7(\lambda) = 0 .
\end{aligned} \tag{10}$$

In the case of a homogeneously magnetized medium ($\mu = 1$), equations (5)-(10) go over into the corresponding equations of [4].

From equations (5)-(10) it can be seen that there are substantial differences between the sonic waves under consideration and waves in a nonmagnetic medium.

In the first place, in an entropy wave, not only ρ and s undergo perturbations, but also the transverse component of the field B_y, so that this wave is not now purely longitudinal.

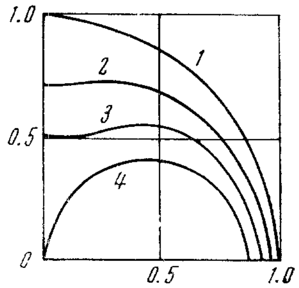

Fig. 1

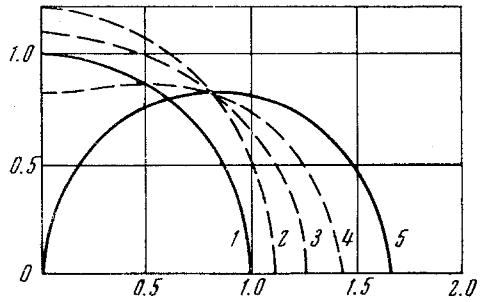

Fig. 2

In the second place, the phase $\lambda_{+}^{(4,5)}$ and $\lambda_{-}^{(6,7)}$ of fast and slow magnetosonic waves in a magnetic substance may take on purely imaginary values, and consequently, in a magnetizable medium there is the possibility of the development of instability similar to that which was investigated [5] in a plasma with transverse and longitudinal pressure. Under these circumstances system (4) becomes partially elliptical, and the motions of a medium in this state of instability obviously cannot be described by system (1).

The phenomenon of instability in a magnetizable medium may arise if any one of the coefficients C_1, C_2 in Eq. (7) becomes negative. We note that in the case of a nonmagnetic medium ($\mu = 1$) we have

$$2\,C_1 = A^2 + a^2 > 0, \quad C_2 = A_x{}^2 a^2 \geqslant 0 \qquad (A^2 = B^2/4\pi\rho\mu)$$

so that, in ordinary magnetic hydrodynamics, the phenomenon of instability could not be recorded.

Let us give examples.

As the first illustration of the propagation of sound waves in a magnetizable medium, let us consider an ideal nonconducting gas ($\mathrm{rot}\,\mathbf{H} = 0$) gas which, in weak magnetic fields, satisfies the equations

$$(\mu - 1)/\mu = C\rho/T \qquad \text{(Mossoti equation)}$$
$$p = \rho RT, \qquad s = c_v \ln(p/\rho^{\varkappa}) \quad (\varkappa = c_p/c_v,\ R = c_p - c_v)\ .$$

For such a medium, by virtue of general equations (5), for components of the matrix $\|x_{ik}\|$ differing from zero we obtain

$$x_{14} = \rho, \quad x_{23} = -\frac{(\mu - 1)\,B_x B_y}{4\pi\rho\mu T\,(1 + \beta)}$$
$$x_{24} = \frac{(\mu - 1)\,B^2\,(1 - \varkappa + \sin^2\vartheta)}{4\pi\rho\mu T\,(1 + \beta)}, \qquad x_{41} = \frac{a^2}{\rho}$$
$$x_{42} = (\varkappa - 1)T, \qquad x_{45} = -(\mu - 1)\,B_y\,/\,4\pi\rho\mu,\ x_{53} = -B_x, \qquad x_{54} = B_y$$

where

$$a^2 = p_\rho = \varkappa p\,/\,\rho, \qquad B_y = B\sin\vartheta$$
$$\alpha = H\,/\,a\sqrt{4\,\pi\rho}, \qquad \beta = \varkappa\mu\,(\varkappa - 1)\,(\mu - 1)\,\alpha^2.$$

Here, since $\mathrm{rot}\,\mathbf{H} = 0$, $v_z' = B_z' = 0$, and the matrix $\|x_{ik}\|$ becomes a matrix of the fifth order. By virtue of (8), we have

$$2C_1 = a^2\left[1 - \frac{\beta\,(2 + \beta - \varkappa)}{\varkappa\,(\varkappa - 1)\,(1 + \beta)}\left(\frac{(\varkappa - 1)^2}{2 + \beta - \varkappa} + \sin^2\vartheta\right)\right], \quad C_2 = 0\ .$$

In the example under consideration the characteristic equation

$$\lambda^3\,(\lambda^2 - 2\,C_1) = 0$$

has three null roots, $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = 0$, and the three linearly independent right-hand vectors corresponding to these roots have the form

$$\mathbf{r}^{(1)} = (-\rho\,/\,\varkappa c_v,\ 1,\ 0,\ 0,\ 0)$$
$$\mathbf{r}^{(2)}\ (\ (\mu - 1)\,B_y\,/\,4\,\pi\mu a^2,\ 0,\ 0,\ 0,\ \ 1)$$
$$\mathbf{r}^{(3)} = (0,\ (\mu - 1)\,B_y\,/\,4\,\pi\mu\,(\varkappa - 1)\,\rho T,\ 0,\ 0,\ 1)$$

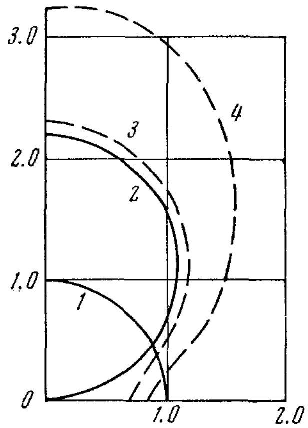

Fig. 3

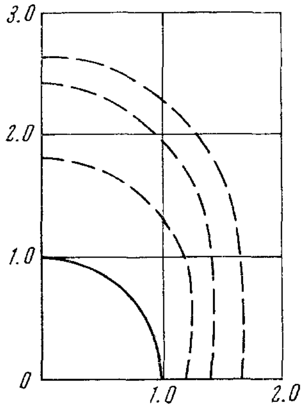

Fig. 4

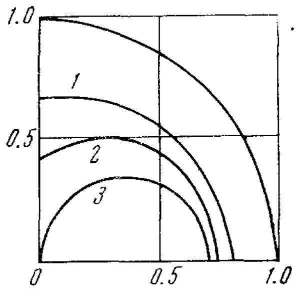

Fig. 5

so that in these waves, which are fixed with respect to the gas, the density, the entropy, and the transverse field B_y vary.

The remaining two characteristic numbers

$$\lambda^{(4,5)} = \pm a\left[1 - \frac{\beta(\beta+2-\varkappa)}{\varkappa(\varkappa-1)(1+\beta)}\left(\frac{(\varkappa-1)^2}{\beta+2-\varkappa} + \sin^2\vartheta\right)\right]^{1/2} \tag{11}$$

correspond to waves whose propagation velocity depends on the orientation of the wave front with respect to the field. This dependence is different for paramagnetic ($\mu > 1$) and diamagnetic ($\mu < 1$) substances.

Typical diagrams of phase velocities for paramagnetic substances are shown on Fig. 1, where curves 1-4 correspond to values of $\beta = 0.0, 0.3, 0.5, 0.655$. Here and in what follows in plotting phase diagrams it is assumed that $\varkappa = 1.4$, and only the first quadrant of the diagrams is given, since they are symmetrical with respect to the axis $\vartheta = 0$, along which the magnetic field is oriented, and to the axis $\vartheta = \pi/2$. As follows from (11), a magnetized paramagnetic substance is stable ($\lambda^{(4,5)}$ are real for any values of ϑ) only in the case

$$B \leqslant \sqrt{\frac{4\pi\mu p f_1(\varkappa)}{\mu - 1}} \qquad \left(f_1(\varkappa) = \frac{1}{\varkappa - 1}\left[\frac{2\varkappa - 3}{2} + \sqrt{\left(\frac{2\varkappa-3}{2}\right)^2 + \varkappa(\varkappa-1)}\right]\right).$$

Thus, if it is assumed for air that $\mu - 1 \approx 10^{-5}$, $p \approx 10^6$ dyn/cm^2, $\varkappa = 1.4$, such a state of the magnetization is stable in fields $B \lessapprox 10^6$ G.

If an ideal gas has the properties of a diamagnetic substance ($\mu < 1$), its state is always stable in the above sense so long as $\alpha^2 \leq 4f_2(\varkappa)$, where

$$f_2(\varkappa) = \frac{1}{\varkappa(\varkappa-1)}\left[\frac{3-2\varkappa}{2} + \sqrt{\left(\frac{3-2\varkappa}{2}\right)^2 + \varkappa(\varkappa-1)}\right].$$

In the case $\alpha^2 > 4f_2(\varkappa)$, the state is stable if

$$2\mu \leqslant \left(1 - \sqrt{1 - 4\alpha^{-2}f_2(\varkappa)}\right), \quad 2\mu \geqslant \left(1 + \sqrt{1 - 4\alpha^{-2}f_2(\varkappa)}\right). \tag{12}$$

Typical diagrams of phase velocities for diamagnetic substances are shown in Fig. 2, where curves 1-5 correspond to values of $\beta = 0, -0.5, -0.7, -0.8, -0.855$.

The dotted curves are characteristic diagrams for the case $\alpha^2 \leq 4f_2(\varkappa)$ and for a value of μ satisfying inequalities (12), while the solid curve, i.e., the semicircle, gives the diagram for $2\mu = 1 \pm \sqrt{1 - 4\alpha^{-2}f_2(\varkappa)}$. Thus, in this case, in a stable state the velocity of the propagation of sound waves may be greater than the velocity of sound in a nonmagnetic gas (curve 1 for $\beta = 0$).

It is interesting that with a very sharply expressed diamagnetism, when $\beta \leq -\varkappa$ and $\alpha^2 \geq 4/(\varkappa - 1)$, the velocity of the propagation of waves across the field becomes greater than the longitudinal velocity and rises in an unlimited manner with an increase in $|\beta|$. Characteristic diagrams for this case are given in Fig. 3, where curves 1-4 correspond to values of $\beta = 0, -\varkappa, -2, -5$.

The type of waves under consideration, propagating with velocities $\lambda^{(4,5)}$, correspond to perturbations described by the vector

$$\mathbf{r}^{(4,5)} = (\rho,\ x_{24},\ 0,\ \lambda^{(4,5)},\ B_y) \tag{13}$$

so that in these waves only the transverse velocity v_y does not vary.

Let us consider a second example. If an ideal nonconducting gas can be magnetized up to a state of saturation, so that the magnetization M ceases to depend on the field, and if it is possible to assume a linear dependence of M on ρ and T, i.e.,

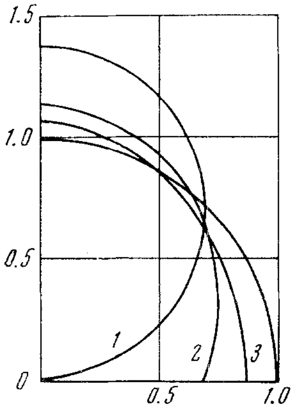

Fig. 6

$$M = \rho K (\theta - T),$$

$$\mu = 1 + 4\pi\rho K (\theta - T) / H \quad (\theta > T),$$

then, from (5) we obtain components of the matrix $\|x_{ik}\|$ differing from zero

$$x_{14} = \rho, \quad x_{23} = -\frac{(\mu - 1) B_x B_y}{4\pi\rho\mu\theta (1 - \tau)(1 - \beta')}$$
$$x_{24} = \frac{(\mu - 1) B^2}{4\pi\rho\theta\, \mu(1 - \tau)(1 - \beta')} \left[\frac{(\mu - 1)(\varkappa\tau - 1)}{\mu (1 - \tau)} + \sin^2 \vartheta\right] \tag{14}$$
$$x_{41} = \frac{a^2}{\rho} + \frac{(\mu - 1)^2 (1 - \varkappa\tau) B^2}{4\pi\rho^2\mu^2 (1 - \tau)}, \quad x_{42} = (\varkappa - 1) T - \frac{(\mu - 1)^2 \tau B^2}{4\pi\rho\mu^2 c_v \theta (1 - \tau)}$$
$$x_{45} = -B_y (\mu - 1)/4\pi\rho\mu, \quad x_{53} = -B_x, \quad x_{54} = B_y$$

where

$$\tau = T / \theta, \; \beta' = \varkappa (\varkappa - 1) (\mu - 1)^2 \tau^2 (1 - \tau)^{-2} \alpha^2 .$$

As in the case of a magnetic substance, in weak fields the triple root $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = 0$ of the characteristic equation corresponds to three linearly independent right-hand vectors, which now have the form

$$r^{(1)} = (-x_{42} / x_{41}, 1, 0, 0, 0)$$
$$r^{(2)} = (0, -x_{45} / x_{42}, 0, 0, 1), \quad r^{(3)} = (x_{45} / x_{41}, 0, 0, 0, 1)$$

where the expressions for x_{41}, x_{42}, x_{45}, correspond to those written in (14).

The remaining two eigenvectors are also the same as in (13), but the expression for x_{24} must be taken from formula (14), while the phase velocities have the form

$$\lambda^{(4,5)} = \pm a \left[1 + \frac{\beta' (\varkappa\tau - 1) \mu (1 - \tau)}{(1 - \beta')(\mu - 1) \varkappa (\varkappa - 1) \tau^2} \left(\frac{(\mu - 1)(\varkappa\tau - 1)}{\mu (1 - \tau)} + \sin^2 \vartheta\right)\right]^{1/2} .$$

The corresponding phase diagrams for a saturated magnetic substance are given in Figs. 4-6.

If $\varkappa\tau > 1$, a saturated magnetic substance is stable with

$$4\pi\rho K\theta / H \leqslant (\alpha\tau)^{-1} [\varkappa (\varkappa - 1)]^{-1/2}$$

(see dotted curves on Fig. 4) and with

$$4\pi\rho K\theta / H \geqslant (\varkappa\tau - 1)(1 + \sqrt{1 + 4\tau (\varkappa - 1) \alpha^{-2}}) [2\tau (\varkappa - 1)]^{-1}$$

(see curves 1-3 on Fig. 5; here curve 1 corresponds to $4\pi\rho K\theta/H = \infty$, and curve 3 to the limiting value of this quantity from the interval of stability). In the first case, the velocity of the propagation of sound waves is greater than in a nonmagnetic gas; here, if $\rho KT = p^{1/2} [4\pi(\varkappa - 1)]^{-1/2}$, the velocity of sound becomes infinitely great with a stable state of the gas. With $p \approx 10^6$ dyn/cm^2, $T \approx 300°$, $\varkappa \approx 1.4$, in this case $\rho K \approx 10$ G/deg, which exceeds by almost three orders of magnitude the corresponding characteristic of artificial ferromagnetic liquids [6].

In the case $\varkappa\tau < 1$ with $4\pi\rho K\theta/H \leq (\alpha\tau)^{-1}[\varkappa(\varkappa - 1)]^{-1/2}$, a magnetic substance is stable if $\alpha \leq \alpha^*$, and with

$$4\pi\rho K\theta / H \leqslant (1 - \varkappa\tau)(1 + \sqrt{1 + 4\tau (\varkappa - 1) \alpha^{-2}}) [2\tau (\varkappa - 1)]^{-1}$$

if

$$\alpha > \alpha^* \quad (\alpha^* = (\varkappa - 1)^{1/2} \varkappa^{-1/2} (1 - \varkappa\tau)^{-1} [1 - \varkappa\tau (1 - \varkappa\tau)^2])$$

so that the diagram of the phase velocities is analogous in form to the diagram on Fig. 4.

For value of $1 > \varkappa\tau \geq (2 - \tau)^{-1}$, this is the interval of stability for large values of the magnetization, that is,

$$4\pi\rho K\theta / H \geqslant \alpha^{-1} [\varkappa\tau (2 - \tau) - 1]^{-1/2} .$$

The diagram for the phase velocities for this case is given on Fig. 6 (see curves 1-3; curve 3 corresponds to $4\pi\rho K\theta/H = \infty$, and curve 1 to the limiting value of this quantity from the interval of stability under consideration).

Finally, we note the case of the propagation of small perturbations in an incompressible liquid. The corresponding equations for this case can be obtained from (7)-(10) by a limiting transition with $p_\rho \equiv a^2 \to \infty$.

The characteristic equation (for finite values of λ) assumes the form

$$\lambda\,(\lambda^2 - B_x^2 / 4\pi\rho\mu)\,(\lambda^2 - x_{23}x_{32} + B_x x_{35}) = 0\,.$$

Thus, in an incompressible nonconducting medium ($x_{32} = x_{35} = x_{67} = 0$), except for entropy waves, there are no other small-amplitude waves propagating with a finite velocity.

In a nonmagnetic conducting liquid, together with purely transverse Alfvén waves, there exist other waves which cannot be called purely longitudinal or purely transverse. Their velocity and the corresponding right-hand eigenvector have the form

$$\lambda = \pm\sqrt{x_{23}x_{32} - B_x x_{35}}\,, \qquad \mathbf{r}^{(\lambda)} = (x_{23}, \lambda, -B_x, 0, 0)\,.$$

Here the components of the matrix $||x_{ik}||$ can be obtained from the general expressions (5) if the equation of state of the liquid at $T = T(s)_*$ is known.

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